NUMERICAL APPROXIMATION OF FRESNEL INTEGRALS BY MEANS OF CHEBYSHEV POLYNOMIALS

by

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Introduction

In this paper a method for computing the Fresnel integrals over the whole range from 0 to ∞ with an accuracy of thirteen decimal places is presented. In section 1 approximations to the integrals are derived in the form of finite series of Chebyshev polynomials, valid in the subranges $0 \leqq x \leqq 5$ and $x \geqq 5$. The coefficients in these approximations have been computed with the aid of a method given by Clenshaw [1]. This method leads to a difference equation from which the coefficients are to be calculated. Section 2 contains an investigation into the errors of the present approximations. In the sections 3 and 4 the third-order difference equation which occurs in the case $x \geqq 5$ has been investigated. In particular it is shown that the solution which was obtained in section 1 really corresponds to the Chebyshev coefficients of the Fresnel integrals.

1. *Derivation of the approximations*

First we quote the following definitions, notations and properties related to the shifted Chebyshev polynomials from Clenshaw [1], with some modifications.

Every function f(x) which is continuous and of bounded variation in $0 \leqq x \leqq 1$ can be expanded in a uniformly convergent series

$$f(x) = \tfrac{1}{2}a_0 + a_1 T_1^*(x) + a_2 T_2^*(x) + \dots = \sum_{r=0}^{\infty}{}' \, a_r T_r^*(x) \qquad (1.1)$$

where $T_r^*(x)$ stands for the shifted r^{th} Chebyshev polynomial defined by

$$T_r^*(x) = \cos r\varphi; \; 2x - 1 = \cos\varphi \qquad \text{for } 0 \leqq x \leqq 1. \qquad (1.2)$$

The prime on the summation symbol in (1.1) and elsewhere in this paper denotes that the term with suffix r = 0 is to be halved.

The orthogonal property of integration of the Chebyshev polynomials gives rise to the following representation of the coefficient a_r in (1.1),

$$a_r = \frac{2}{\pi} \int_0^1 f(x) \frac{T_r^*(x)}{\sqrt{x-x^2}} \, dx \qquad (r = 0, 1, 2, \dots), \qquad (1.3)$$

the so-called r^{th} Fourier-Chebyshev coefficient of the function f(x).

The r^{th} Fourier-Chebyshev coefficient of the s^{th} derivative, $f^{(s)}(x)$, of a function f(x) is denoted by $a_r^{(s)}$. By means of an integration by parts, one can derive the relation

$$4ra_r^{(s)} = a_{|r-1|}^{(s+1)} - a_{r+1}^{(s+1)} \qquad (1.4)$$

under the conditions

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$$f^{(s)}(x) = o\,(x^{-\frac{1}{2}}) \qquad \text{as } x \to 0,$$

$$f^{(s)}(x) = o\,((1-x)^{-\frac{1}{2}}) \text{ as } x \to 1.$$

Let the r^{th} Fourier-Chebyshev coefficient of the function $x^p f^{(s)}(x)$ be denoted by $C_r(x^p f^{(s)})$, then one can easily derive that

$$C_r(xf^{(s)}) = \tfrac{1}{4}\,(a^{(s)}_{|r-1|} + 2\,a^{(s)}_r + a^{(s)}_{r+1}) \tag{1.5}$$

and hence

$$C_r(x^p f^{(s)}) = 2^{-2p} \sum_{j=0}^{\infty} \binom{2p}{j} a^{(s)}_{|r-p+j|} \quad (p = 0, 1, 2, \ldots), \tag{1.6}$$

under no other conditions than the existence of the occurring coefficients.

In this paper the Fresnel integrals are defined by

$$C(x) = \int_0^x \frac{\cos t}{\sqrt{2\pi t}}\,dt\,,$$
$$S(x) = \int_0^x \frac{\sin t}{\sqrt{2\pi t}}\,dt\,. \tag{1.7}$$

Approximations in the form of finite Chebyshev series expansions will be derived for the following function,

$$f(x) = \int_0^x \frac{e^{-it}}{\sqrt{2\pi t}}\,dt = C(x) - iS(x)\,. \tag{1.8}$$

Two different expansions will be obtained, valid for the ranges $0 \leqq x \leqq 5$ and $x \geqq 5$, respectively.

At first we construct the approximation for $0 \leqq x \leqq 5$. If we expand the function e^{-it}, a term by term integration of (1.8) leads to the following expansion for the function f(x),

$$f(x) = \sqrt{\frac{x}{2\pi}} \sum_{r=0}^{\infty} \frac{(-ix)^r}{r!\,(r+\frac{1}{2})}. \tag{1.9}$$

Now we introduce a function u(x) defined by

$$f(x) = \sqrt{\frac{x}{2\pi}}\,u(x)\,. \tag{1.10}$$

Inspection of (1.9) shows that the function u(x) is more suited to approximation by polynomials.

Differentiation of (1.10) and subsequent substitution of x=5z yield the following differential equation for u as a function of z,

$$z\,\frac{du}{dz} + \frac{u}{2} = e^{-5iz} \qquad (0 \leqq z \leqq 1). \tag{1.11}$$

The right-hand side of (1.11) can be expanded in a Chebyshev series,

$$e^{-5iz} = e^{-\frac{5}{2}i(2z-1)}e^{-\frac{5}{2}i} = e^{-\frac{5}{2}i} \cdot 2 \sum_{r=0}^{\infty}{}' (-i)^r J_r(\tfrac{5}{2}) T_r^*(z), \tag{1.12}$$

where J_r denotes the Bessel function of the first kind. Relation (1.12) can easily be deduced from the expansion

$$e^{ia\cos\varphi} = 2\sum_{r=0}^{\infty}{}' i^r J_r(a) \cos r\varphi, \quad ([2], \text{ form. } 7.2\ (27)).$$

Hence we obtain the following relation between Fourier-Chebyshev coefficients,

$$C_r(z\,u') + \tfrac{1}{2} C_r(u) = 2\, e^{-\frac{5}{2}i} (-i)^r J_r(\tfrac{5}{2}) \quad (r=0,1,2,\dots). \tag{1.13}$$

When we use (1.5), the relation (1.13) becomes

$$a'_{|r-1|} + 2\,a'_r + a'_{r+1} + 2\,a_r = 8\, e^{-\frac{5}{2}i} (-i)^r J_r(\tfrac{5}{2}). \tag{1.14}$$

The latter relation combined with (1.4) will be used for the computation of the coefficients a_r of the truncated Chebyshev series expansions for u(z). We eliminate the coefficients a'_r by first subtracting the relation (1.14) with r replaced by r+1 from the original relation (1.14). This leads to

$$a'_{|r-1|} + a'_r - a'_{r+1} - a'_{r+2} + 2\,a_r - 2\,a_{r+1} = 8\, e^{-\frac{5}{2}i} (-i)^r \{J_r(\tfrac{5}{2}) + i\, J_{r+1}(\tfrac{5}{2})\}. \tag{1.15}$$

Using (1.4) we can eliminate the coefficients a'_r and we obtain the following difference equation for the coefficients a_r,

$$a_r + a_{r+1} = \frac{4e^{-\frac{5}{2}i}\, (-i)^r \{J_r(\frac{5}{2}) + i\, J_{r+1}(\frac{5}{2})\}}{2r+1}. \tag{1.16}$$

Since the Chebyshev series formed with the coefficients a_r should converge, equation (1.16) must be solved with the boundary condition

$$\lim_{r\to\infty} a_r = 0\,. \tag{1.17}$$

If the calculation is performed in 14 decimal places, it appears that the right-hand side of equation (1.16) vanishes in this accuracy for $r \geq 18$. This means that all coefficients a_r with $r \geq 18$ can be taken as zero. Then the coefficients for $r \leq N = 17$ can be calculated from equation (1.16) directly. The stability of the procedure is satisfactory since an error in one of the coefficients a_r is transferred with equal absolute value to all coefficients with smaller r. However, the solution obtained increases rapidly with smaller r.

Secondly we construct the polynomial approximation for $x \geq 5$. It can be derived, that

$$f(\infty) = \int_0^{\infty} \frac{e^{-it^2}}{\sqrt{2\pi t}}\, dt = \frac{1-i}{2}\,.$$

By means of repeated integration by parts, we derive

$$\int_x^\infty \frac{e^{-it}}{\sqrt{2\pi t}}\,dt \sim \frac{-ie^{-it}}{\sqrt{2\pi x}} \sum_{r=0}^{\infty} \frac{\Gamma(\frac{1}{2}+r)}{\Gamma(\frac{1}{2})} \left(\frac{i}{x}\right)^r . \tag{1.18}$$

The asymptotic approximation (1.18) shows that the function f(x) is not suited to approximation by polynomials in x^{-1}. Therefore, we introduce a function u(x), defined by

$$f(x) = \frac{1-i}{2} - \frac{e^{-ix}}{\sqrt{2\pi x}}\, u(x)$$

or

$$u(x) = e^{ix}\sqrt{x} \int_x^\infty \frac{e^{-it}}{\sqrt{t}}\,dt . \tag{1.19}$$

Differentiation of (1.19) and subsequent substitution of $x=5/z$ yield the following differential equation for u as a function of z,

$$z^2 \frac{du}{dz} + u\left(\tfrac{1}{2} z + 5\, i\right) = 5 \quad (0 \leqq z \leqq 1). \tag{1.20}$$

From (1.20) we obtain the following relations between Fourier-Chebyshev coefficients,

$$\begin{cases} C_r(z^2u') + \frac{1}{2}\, C_r(zu) + 5\, i\, C_r(u) = 0 & (r=1,2,\ldots), \\ C_0(z^2u') + \frac{1}{2}\, C_0(zu) + 5\, i\, C_0(u) = 10 & (r=0). \end{cases} \tag{1.21}$$

When we use (1.6), the relations (1.21) become

$$\begin{cases} a'_{|r-2|} + 4\,a'_{r-1} + 6\,a'_r + 4\,a'_{r+1} + a'_{r+2} + 2\,a_{r-1} + 4\,a_r + 2\,a_{r+1} + 80\, ia_r = 0 \\ \qquad (r=1,2,\ldots), \\ 6\,a'_0 + 8\,a'_1 + 2\,a'_2 + 4\,a_1 + 4\,a_0 + 80\, i\, a_0 = 160 \quad (r=0). \end{cases} \tag{1.22}$$

We subtract the relations (1.22) with r replaced by r+1 from the original relations (1.22) and use (1.4), in order to eliminate the coefficients a'_r. The result is

$$\begin{cases} (2r-1)\, a_{r-1} + (6r+1+40\,i)\, a_r + (6r+5-40\,i)\, a_{r+1} + (2r+3)\, a_{r+2} = 0 \\ \qquad (r=1,2\ldots), \\ (1+40i)\, a_0 + (4-40i)\, a_1 + 3\, a_2 = 80 \qquad (r=0). \end{cases} \tag{1.23}$$

Equation (1.23) for $r \geq 1$ is a third-order difference equation. Two boundary conditions are available, viz. equation (1.23) for r=0 and condition (1.17). This, however, is not sufficient to determine the solution uniquely. One solution has been obtained in the following way.

For a sufficiently large integer M we assume

$$a_M = 1,\; a_{M+1} = a_{M+2} = \ldots = 0 .$$

Then the coefficients a_r for $r<M$ can be solved from equation (1.23) for $r=M, M-1, \ldots, 1$. Equation (1.23) for r=0 acts as a normalizing relation, by means of which final values for all coefficients can be obtained. It was found that for increasing M the values of the coefficients a_r converge. For $M \geqq 19$ these coefficients are obtained with an accuracy of fourteen decimal places. It appears that a_{19} vanishes in this accuracy so that only

the coefficients a_r $(r=0, 1, \ldots, N)$ with $N=18$ are of importance.

In section 3 and 4 an investigation of the solutions of equation (1.23) has been carried out. In particular, it is shown there that the coefficients obtained above correspond to the coefficients of the Chebyshev series expansion of the function u(x) as defined by relation (1.19).

Thus we have obtained the following approximations,

$$f(x) \approx \sqrt{\frac{x}{2\pi}} \sum_{r=0}^{N}{}' a_r T_r^*\left(\frac{x}{5}\right) \text{ valid for } 0 \leqq x \leqq 5 \tag{1.24}$$

and

$$f(x) \approx \frac{1-i}{2} - \frac{e^{-ix}}{\sqrt{2\pi x}} \sum_{r=0}^{N}{}' a_r T_r^*\left(\frac{5}{x}\right) \text{ valid for } x \geqq 5. \tag{1.25}$$

Actually, the coefficients a_r in (1.24) and (1.25) have been multiplied by $\sqrt{\frac{5}{2\pi}}$ and $\frac{1}{\sqrt{10\pi}}$, respectively, in order to obtain the approximations

$$f(x) \approx \sqrt{\frac{x}{5}} \sum_{r=0}^{N}{}' (a_{1r} + i\, a_{2r})\, T_r^*\left(\frac{x}{5}\right) \text{ for } 0 \leqq x \leqq 5 \tag{1.26}$$

and

$$f(x) \approx \frac{1-i}{2} - e^{-ix}\sqrt{\frac{5}{x}} \sum_{r=0}^{N}{}' (a_{3r} + i\, a_{4r})\, T_r^*\left(\frac{5}{x}\right) \text{ for } x \geqq 5. \tag{1.27}$$

The computations of the coefficients a_{kr} in the approximations (1.26) and (1.27) have been performed on the digital computer ZEBRA operating with numbers of 16 decimal digits (the so-called $1\frac{1}{2}$ length numbers). In this paper the computed coefficients are given in 14 decimal places in Table 1. The inaccuracy of these coefficients will be one unit of the last decimal at most.

TABLE 1

r	a_{1r}		a_{2r}	
0	+ 2.0317861	9253011	- 1.1205451	1759094
1	- .8319294	4359172	- .1971640	8056849
2	+ .0530351	6304029	+ .3533223	3350780
3	+ .1094828	9102595	- .0389725	5167861
4	- .0132731	4036389	- .0269745	9917782
5	- .0055349	7821362	+ .0032735	0051478
6	+ .0006511	1454310	+ .0009744	2919474
7	+ .0001503	0914539	- .0001096	9214833
8	- . 161	0750990	- . 206	3782833
9	- . 25	5346638	+ . 21	0172858
10	+ . 2	4711976	+ . 2	8753088
11	+ .	2970767	- .	2646656
12	- .	260422	- .	283565
13	- .	25152	+ .	23709
14	+ .	2009	+ .	2083
15	+ .	162	- .	159
16	- .	12	- .	12
17	- .	1	+ .	1

r	a_{3r}		a_{4r}	
0	+ .0166129	7830452	- .3534565	6795162
1	+ .0080271	5445291	+ .0021736	8937097
2	- .0003129	8659925	+ .0004429	5213449
3	- . 261	4348478	- . 474	7464222
4	+ . 69	8182019	+ . 1	9251930
5	- . 6	2323535	+ . 8	7130944
6	- .	5626289	- . 1	7824386
7	+ .	3348055	+ .	1339747
8	- .	673601	+ .	340402
9	+ .	47556	- .	162191
10	+ -	19414	+ .	35080
11	- .	9554	- .	2940
12	+ .	2347	- .	1125
13	- .	273	+ .	650
14	- .	61	- .	188
15	+ .	48	+ .	31
16	- .	17	+ .	2
17	+ .	4	- .	4
18			+ .	2

For the convenience of the user we provide the following check sums,

$$\Sigma' a_{1r} = +0.3284566 \quad 2486755,$$

$$\Sigma' (-1)^r a_{1r} = \sqrt{\frac{10}{\pi}} = +1.7841241 \quad 1615277,$$

$$\Sigma' a_{2r} = -0.4659414 \quad 9676626,$$

$$\Sigma' (-1)^r a_{2r} = 0,$$

$$\Sigma' a_{3r} = +0.0160008 \quad 4318228,$$

$$\Sigma' (-1)^r a_{3r} = 0,$$

$$\Sigma' a_{4r} = -0.1741582 \quad 1603304,$$

$$\Sigma' (-1)^r a_{4r} = \frac{-1}{\sqrt{10\pi}} = -0.1784124 \quad 1161528.$$

2. *Discussion of the errors.*

There are three sources of errors for the final values of the Fresnel integrals, viz.

(i) The Chebyshev series have been truncated.

(ii) The coefficients in the Chebyshev series have been rounded to 14 decimal places.

(iii) The method of evaluating $\sum_{r=0}^{N}{}' a_r T_r(x)$ as given by Clenshaw [1], may introduce an additional error.

Ad (i). The truncation in the Chebyshev series is due to the calculation of the coefficients a_r. which implies the choice of a number N and the assumption that $a_{N+1} = a_{N+2} = \dots = 0$ (cf. (1.17) and (1.23)). Now we shall investigate the truncation errors for the two approximations.

1. For the range $0 \leq x \leq 5$ the original function u(z) satisfies the differential equation (1.11). The approximation to this function u(z), called $u_N(z)$, satisfies the differential equation (cf. (1.12))

$$z\, u_N' + \frac{u_N}{2} = 2\, e^{-\frac{5}{2}i} \sum_{r=0}^{N}{}' (-1)^r J_r\left(\frac{5}{2}\right) T_r^*(z). \tag{2.1}$$

The error $\eta_N(z) = u(z) - u_N(z)$ will therefore satisfy the equation

$$z\,\eta_N' + \frac{\eta_N}{2} = 2\,e^{-\frac{5}{2}i} \sum_{r=N+1}^{\infty} (-i)^r\, J_r\left(\tfrac{5}{2}\right) T_r^*(z). \tag{2.2}$$

The latter differential equation can be integrated, which yields

$$\eta_N(z) = \frac{2e^{-\frac{5}{2}i}}{\sqrt{z}} \sum_{r=N+1}^{\infty} (-i)^r\, J_r\left(\tfrac{5}{2}\right) \int_0^z \frac{T_r^*(t)}{\sqrt{t}}\, dt. \tag{2.3}$$

The error $\epsilon_N(x)$ in the corresponding approximation (1.24) to the Fresnel integrals can be represented by

$$\epsilon_N(x) = \sqrt{\frac{10}{\pi}}\, e^{-\frac{5}{2}i} \sum_{r=N+1}^{\infty} (-i)^r\, J_r\left(\tfrac{5}{2}\right) \int_0^{x/5} \frac{T_r^*(t)}{\sqrt{t}}\, dt. \tag{2.4}$$

The representation (2.4) can be estimated in the following manner. When we substitute $t=u^2$, the integral in the right-hand side of (2.4) changes into

$$\int_0^{x/5} \frac{T_r^*(t)}{\sqrt{t}}\, dt = 2 \int_0^{\sqrt{x/5}} T_r^*(u^2)\, du = 2 \int_0^{\sqrt{x/5}} T_{2r}(u)\, du \tag{2.5}$$

$$= \frac{T_{2r+1}(\sqrt{x/5})}{2r+1} - \frac{T_{2r-1}(\sqrt{x/5})}{2r-1},$$

according to [1], form. (11) and (5). On the range $0 \leqq x \leqq 5$ the integral (2.5) assumes a maximum or a minimum for

$$x = x_k = 5\cos^2 \frac{(2k+1)\pi}{4r} \qquad (k=0, 1, \ldots, r-1),$$

the values x_k being the zeros of the function $T_r^*(x/5)$. The actual maxima and minima are given by

$$\frac{T_{2r+1}(\sqrt{x_k/5})}{2r+1} - \frac{T_{2r-1}(\sqrt{x_k/5})}{2r-1} = (-1)^{k+1} \frac{4r}{4r^2-1} \sin(2k+1)\frac{\pi}{4r}. \tag{2.6}$$

Hence we can estimate

$$\left| \int_0^{x/5} \frac{T_r^*(t)}{\sqrt{t}}\, dt \right| \leqq \frac{4r}{4r^2-1},$$

$$\left| \epsilon_N(x) \right| \leqq \sqrt{\frac{10}{\pi}} \sum_{r=N+1}^{\infty} \frac{4r}{4r^2-1} J_r\left(\tfrac{5}{2}\right), \tag{2.7}$$

both valid for $0 \leqq x \leqq 5$. As was mentioned in section 1 N has been taken equal to 17. By means of a computed table of the Bessel functions $J_r(\frac{5}{2})$ we found the following upper bound for the error $\epsilon_{17}(x)$,

$$\left| \epsilon_{17}(x) \right| < 0.85 \times 10^{-15}.$$

This error is negligible in comparison with the errors due to (ii) and (iii).

2. For the range $x \geqq 5$ the original function u(z) satisfies the differential equation (1.20). Now we shall derive a differential equation for the approximation to this function u(z), called $u_M(z)$. We denote the relations (1.22) by

$$F_r = 0 \qquad (r=0,1,2,\ldots).$$

Instead of these relations we solved

$$F_r - F_{r+1} = 0 \qquad (r=0,1,2,\ldots,M)$$

or

$$F_r = \tau \qquad (r=0,1,2,\ldots,M+1), \tag{2.8}$$

where τ is some constant.
From (1.22) with r=M+1 and (1.4) it follows that

$$\tau = a'_{M-1} + 2\,a_M = (4\,M+2)\,a_M, \tag{2.9}$$

as the higher order coefficients have been taken zero. Consequently the approximation $u_M(z)$ is a solution of the equation

$$z^2 u'_M + \tfrac{1}{2} z\, u_M + 5\,i\,u_M = 5 + \frac{2\,M+1}{8}\, a_M \sum_{r=0}^{M+1}{}' \; T^*_r(z). \tag{2.10}$$

According to [1], form. (11) and to [2], form. 10.11 (40) the sum in the right-hand side of (2.10) can be written as

$$\sum_{r=0}^{M+1}{}' \; T^*_r(z) = \sum_{r=0}^{M+1}{}' \; T_{2r}(\sqrt{z}) = \tfrac{1}{2}\, U_{2M+2}(\sqrt{z}), \tag{2.11}$$

where U_r denotes the Chebyshev polynomial of the second kind. The error $\eta_M(z) = u(z) - u_M(z)$ will now satisfy the equation

$$z^2 \eta'_M + \tfrac{1}{2} z\, \eta_M + 5\,i\,\eta_M = -\frac{2M+1}{16}\, a_M\, U_{2M+2}(\sqrt{z}). \tag{2.12}$$

The latter differential equation can be integrated, which yields

$$\eta_M(z) = -\frac{2\,M+1}{16}\, a_M \frac{e^{5i/z}}{\sqrt{z}} \int_0^z U_{2M+2}(\sqrt{t})\, \frac{e^{-5i/t}}{t\sqrt{t}}\, dt. \tag{2.13}$$

The error $\epsilon_M(x)$ in the corresponding approximation (1.25) to the Fresnel integrals can be represented by

$$\epsilon_M(x) = \frac{1}{\sqrt{10\,\pi}} \frac{2\,M+1}{16}\, a_M \int_0^{5/x} U_{2\,M+2}(\sqrt{t})\, \frac{e^{-5i/t}}{t\,\sqrt{t}}\, dt. \tag{2.14}$$

On the range $x \geqq 5$ the real and the imaginary part of the integral in the right-hand side of (2.14) will both assume a maximum or a minimum for

$$x = x_k = 5 \,/\, \cos^2 \frac{k\,\pi}{2\,M+3} \qquad (k=1,2,\ldots,M+1), \tag{2.15}$$

the values x_k being the zeros of the function $U_{2M+2}(\sqrt{5/x})$. Besides, the real and imaginary parts of the right-hand side of (2.14) will assume maxima and minima for values of x satisfying

$$\operatorname{Re}\{a_M e^{-ix}\} = 0 \text{ and } \operatorname{Im}\{a_M e^{-ix}\} = 0\ , \tag{2.16}$$

respectively.

As was mentioned in section 1, M has been taken equal to 19, which leads to the following numerical value for $a_{19}/\sqrt{10\pi}$,

$$\frac{a_{19}}{\sqrt{10\pi}} = -(0.1171 + 0.5169\, i) \times 10^{-14}. \tag{2.17}$$

For $x \geqq 200$ the representation (2.14) can be estimated with the aid of an integration by parts. The result is

$$|\epsilon_{19}(x)| < 0.25 \times 10^{-14}. \tag{2.18}$$

For $5 \leqq x \leqq 200$ the following bounds were obtained by numerical integration,

$$\left| \operatorname{Re}\left\{ \frac{39}{16\sqrt{10\pi}}\, a_{19} \int_{5/200}^{5/x} U_{40}(\sqrt{t})\, \frac{e^{-5i/t}}{t\sqrt{t}}\, dt \right\} \right| < 0.37 \times 10^{-14}, \tag{2.19}$$

$$\left| \operatorname{Im}\left\{ \frac{39}{16\sqrt{10\pi}}\, a_{19} \int_{5/200}^{5/x} U_{40}(\sqrt{t})\, \frac{e^{-5i/t}}{t\sqrt{t}}\, dt \right\} \right| < 0.78 \times 10^{-14}.$$

Splitting the integral in the right-hand side of (2.14) into two parts

$$\int_0^{5/x} = \int_0^{5/200} + \int_{5/200}^{5/x}\ ,$$

we obtain the following bounds for the errors $\operatorname{Re}\{\epsilon_{19}(x)\}$ and $\operatorname{Im}\{\epsilon_{19}(x)\}$ in the approximations to the Fresnel integrals for $x \geqq 5$ from (2.18) and (2.19),

$$|\operatorname{Re}\{\epsilon_{19}(x)\}| < 0.62 \times 10^{-14},$$

$$|\operatorname{Im}\{\epsilon_{19}(x)\}| < 1.03 \times 10^{-14}.$$

These errors are negligible in comparison with the errors due to (ii) and (iii).

(ii). Rounding the coefficients a_r to 14 decimal places may lead to a rounding error of one unit in the 13th decimal digit in the calculation of $\sum_{r=0}^{N}{}' a_r T_r(x)$.

(iii). It has been shown by Clenshaw [1], §5.1, that the method of evaluation of the series $\sum_{r=0}^{N}{}' a_r T_r(x)$ with the aid of a recurrent relation leads to errors which are of the same order of magnitude as those under (ii).

The final result is that with the data of this paper the Fresnel integrals can be calculated accurate to 13 decimal places.

3. *Properties of the solutions of the system (1.23).*

In section 1 we obtained the recursive system (1.23). The general solution of this system is of the type

$$a_r = K_r + c_1 A_r + c_2 B_r \qquad (r=0,1,2,\ldots), \tag{3.1}$$

where K_r is a particular solution of the system; A_r and B_r are linearly independent solutions of the system when the right-hand side of the second relation is taken zero (the reduced system), c_1 and c_2 are arbitrary constants.

In this section it will be shown that only one solution a_r of the recursive system (1.23) has the property that the series $\sum_{r=0}^{\infty}{}' a_r T_r^*(z)$ converges to a bounded function throughout the range $0 \leq z \leq 1$.

We know that the coefficients in the Chebyshev series expansion of the function $u(5/z)$, as defined in (1.19), constitute a solution of the recursive system. Let K_r in (3.1) denote the r-th order coefficient in this expansion. Then we have to show that $\sum_{r=0}^{\infty}{}' A_r T_r^*(z)$ and $\sum_{r=0}^{\infty}{}' B_r T_r^*(z)$ cannot converge to a bounded function throughout the range $0 \leq z \leq 1$.

A formal application of the method described in section 1 to the reduced form of the differential equation (1.20)

$$z^2 \frac{du}{dz} + u\left(\tfrac{1}{2} z + 5i\right) = 0 \tag{3.2}$$

yields the reduced recursive system. Therefore we investigate whether the Fourier-Chebyshev coefficients of the solution of (3.2) exist and satisfy the reduced system.

Formation of the r^{th} Fourier-Chebyshev coefficient of the solution $u(z) = z^{-\frac{1}{2}} e^{5i/z}$ (cf.(1.3)) yields a convergent integral. Substitution of these coefficients into the equations of the reduced system shows that they constitute a solution of that system.

Let A_r in (3.1) denote the r^{th} order Fourier-Chebyshev coefficient of $z^{-\frac{1}{2}} e^{5i/z}$. It has been proved by Braaksma that the series $\sum_{r=0}^{\infty}{}' A_r T_r^*(z)$ converges to $z^{-\frac{1}{2}} e^{5i/z}$ in any interval $\epsilon \leq z \leq 1$ with $\epsilon > 0$ (cf. [3]). Consequently,

$$|A_r| \to 0 \text{ as } r \to \infty, \tag{3.3}$$

but the series $\sum_{r=0}^{\infty}{}' A_r T_r^*(z)$ does not converge to a bounded function throughout the range $0 \leq z \leq 1$.

A numerical solution of the first relation of (1.23) for r=1 up to r=50 by means of forward recurrence with some starting values for a_0, a_1, a_2 pointed to the existence of a solution P_r with rapidly increasing absolute values for increasing values of r up to r=50.
Since the asymptotic form of the first relation of (1.23) as $r \to \infty$, i.e.

$$a_{r-1} + 3a_r + 3a_{r+1} + a_{r+2} = 0 \,,$$

has $(-1)^r$, $r(-1)^r$ and $r^2(-1)^r$ as linearly independent solutions, it is reasonable to conclude that $|P_r| \not\to 0$ as $r \to \infty$. This means that the solution B_r of the reduced recursive system has the property $|B_r| \not\to 0$ as $r \to \infty$, because $|A_r|$ and $|K_r| \to 0$ as $r \to \infty$. Consequently the series $\sum_{r=0}^{\infty}{}' B_r T_r^*(z)$ will certainly not converge in the range $0 \leq z \leq 1$.

4. *The numerical solutions of the system (1.23).*

In the paragraph following the recursive system (1.23) we described a method, which has been applied in order to obtain a finite number of accurate values for a desired solution of the system. The correctness of the method depends on the behaviour of the linearly independent solutions of the third-order difference equation formed by the first relation of (1.23). In this section the results of a numerical investigation into the behaviour of the solutions of the difference equation will be given. The computations were performed on the Telefunken-TR4 computer of Groningen University in 11 decimal digits.

First the difference equation was solved numerically for $r=1$ up to $r=50$ by means of backward and forward recurrence with the three linearly independent combinations of starting values 1,0,0;0,1,0;0,0,1 for a_{50}, a_{51}, a_{52} and a_0, a_1, a_2, respectively. The backward recurrence yielded three sets of values for a_r, which after a normalization on $a_0 = 1$ turned out to be identical in 10 significant figures for $r=0,1,2,\ldots,25$. The absolute values for a_r in this range increased from $|a_{25}| \approx 2.3 \times 10^{-17}$ to $|a_0| = 1$. The forward recurrence yielded three sets of values for a_r, which after a normalization on $a_{50} = 1$ turned out to be identical in 10 significant figures for $r=25,26,27,\ldots,52$. The absolute values for a_r in this range increased from $|a_{25}| \approx 5.1 \times 10^{-11}$ to $|a_{50}| = 1$. We may conclude that for $r=0,1,2,\ldots,50$ the difference equation has one rapidly decreasing and one rapidly increasing solution, each of them dominating in one of the directions of recurrence.

Next the difference equation was extended to

$$(2r-1)a_{r-1} + (6r+1+40\,i)\,a_r + (6r+5-40\,i)\,a_{r+1} + (2r+3)\,a_{r+2} = 0 \tag{4.1}$$

for all integer values of r.

We remark that if a_r constitutes a solution of the difference equation thus extended, $b_r = a_{-r}$ will also be a solution. This can be shown as follows.

Substitution of $r = -r' - 1$ in equation (4.1) yields

$$(-2r'-3)\,a_{-r'-2} + (-6r'-5+40\,i)\,a_{-r'-1} + (-6r'-1-40\,i)\,a_{-r'} + (-2r'+1)a_{-r'+1} = 0.$$

Defining $b_{r'} = a_{-r'}$ this last relation becomes

$$(2r'+3)\,b_{r'+2} + (6r'+5-40\,i)\,b_{r'+1} + (6r'+1+40\,i)\,b_{r'} + (2r'-1)\,b_{r'-1} = 0\ ,$$

which proves the statement.

Equation (4.1) was solved numerically for $r=50$ up to $r=-49$ by means of backward recurrence with the starting conditions $a_{50}=1$, $a_{51}=a_{52}=0$. The computed values for a_r were normalized on $a_0 = 1$ subsequently. The absolute values increased from $|a_{25}| \approx 2.3 \times 10^{-17}$ to $|a_{-50}| \approx 3.8 \times 10^{25}$. In a number of significant figures this set of values for a_r was assumed to represent a solution of (4.1), the absolute values of which decrease rapidly for values of r increasing from $r=-50$ to $r=+50$. We shall denote this solution by the vector $\mathbf{y}_1$ and its elements by $(y_1)_r$. Due to the symmetry property of the difference equation the vector $\mathbf{y}_2$, defined by $(y_2)_r = (y_1)_{-r}$, will be a solution, the absolute values of which increase rapidly. The linear combinations $\frac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2)$ and $\frac{1}{2}(\mathbf{y}_1 - \mathbf{y}_2)$ will then constitute a symmetrical and an antisymmetrical solution with regard to $r=0$.

Now we look for a third solution of (4.1) linearly independent of the solutions $\mathbf{y}_1$ and $\mathbf{y}_2$. Substitution of $a_{-r} = a_r$ in (4.1) yields

$$\begin{cases}(1+40\,i)\,a_0 + (4-40\,i)\,a_1 + 3\,a_2 = 0, \\ (2r-1)\,a_{r-1} + (6r+1+40\,i)\,a_r + (6r+5-40\,i)\,a_{r+1} + (2r+3)\,a_{r+2} = 0 \end{cases} \tag{4.2}$$

$$(r=1,2,3,\dots).$$

The recursive system (4.2) has two linearly independent solutions. Both solutions will form symmetrical solutions of equation (4.1). Therefore we look for a second symmetrical solution of (4.1), say $\mathbf{y}_3$, the absolute values of which increase at least less rapidly than the absolute values for $\frac{1}{2}(\mathbf{y}_1+\mathbf{y}_2)$.

Equation (4.1) was solved numerically for $r=0$ up to $r=50$ by means of forward recurrence with the starting conditions $a_0=0, a_{-1}=a_1=1$. Because of the starting conditions the set of values thus computed will represent a branch of a symmetrical solution $\mathbf{y}_3'$. Apart from a constant factor the values for $(y_3')_r$ became identical in 10 significant figures with the values for $\frac{1}{2}(y_1+y_2)_r$ for r in the neighbourhood of $r=50$ as was to be expected, since both solutions will have the most rapidly increasing solution as a component. We may expect that a linear combination of $\mathbf{y}_3'$ and $\frac{1}{2}(\mathbf{y}_1+\mathbf{y}_2)$ will increase less rapidly.

Denoting the computed values for $\frac{1}{2}(y_1+y_2)_{50}$ and $(y_3')_{50}$ by p and q, respectively, we may expect that the values for

$$\mathbf{y}_3'' = \mathbf{y}_3' - \frac{q}{p} \cdot \tfrac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2) \tag{4.3}$$

will coincide with the values for the desired solution $\mathbf{y}_3$ in a number of significant figures for r in the neighbourhood of $r=0$.
The values for $\mathbf{y}_3''$ were computed by means of forward recurrence with the starting conditions $a_0=(y_3'')_0$, $a_{-1} = a_{+1} = (y_3'')_1$ derived from (4.3). The computations yielded a set of values for $\mathbf{y}_3''$ with absolute values decreasing from $r=0$ to $r=11$ and increasing for $r>11$. The values for

$$\left|\frac{(y_3'')_{50}}{(y_3'')_{49}}\right| \quad \text{and} \quad \left|\frac{(y_1+y_2)_{50}}{(y_1+y_2)_{49}}\right|$$ were identical again in 10 significant figures, which means that the most rapidly increasing solution had entered in the computations by rounding errors during the process of recurrence. Therefore we repeated the computations by means of forward recurrence, now starting with the conditions $a_9 = (y_3''')_9$, $a_{10} = (y_3''')_{10}$, $a_{11} = (y_3''')_{11}$, where

$$\mathbf{y}_3''' = \mathbf{y}_3'' - \frac{q'}{p} \cdot \tfrac{1}{2}(\mathbf{y}_1 + \mathbf{y}_2)$$

and q' denotes the computed value for $(y_3'')_{50}$. Thus we obtained values for $\mathbf{y}_3$ correct in at least 3 significant figures for $r=0$ to $r=28$. These values are given in Table 2. For $r>29$ the absolute values began to increase again.

The system (4.2) is identical with the reduced form of the recursive system (1.23). Therefore the solutions of (4.2) are linear combinations of the solutions A_r and B_r introduced in section 3. A_r has been defined by

$$A_r = \frac{2}{\pi}\int_0^1 z^{-\frac{1}{2}}\, e^{5iz^{-1}} \frac{T_r^*(z)}{\sqrt{z-z^2}}\, dz \qquad (r=0,1,2,\dots). \tag{4.4}$$

Apart from the property $|B_r| \not\to 0$ as $r \to \infty$, the solution B_r has not been defined. The computations described in this section pointed to the existence of a solution of (4.2) with rapidly increasing absolute values for $r=0$ up to r=50. We denote this latter solution by B_r now. The values for $\mathbf{y}_3$ were computed by means of a proceeding reduction of the component B_r in the solutions $\mathbf{y}_3'$ and $\mathbf{y}_3''$. Consequently, apart from a constant factor, the solution $\mathbf{y}_3$ for $r \geq 0$ can be identified with the solution A_r.

Conclusion. We have found that one of the solutions of the difference equation, formed by the first relation of (1.23), has rapidly increasing absolute values and consequently dominates in the process of forward recurrence. A second solution has rapidly decreasing values and therefore dominates in the process of backward recurrence. The values of a third independent solution have been given in Table 2. This solution has been identified with A_r (4.4), apart from a constant factor. Since the coefficients K_r in the Chebyshev series expansion for the function $u(5/z)$, defined in (1.19), constitute a solution of the difference equation and cannot have a component A_r (cf. section 3), we can identify the second solution with K_r. Consequently, we may conclude that the absolute value of an error introduced in any step of the computation, either by the choice of the starting conditions or by rounding errors, can increase during the process of backward recurrence at most in the same manner as $|K_r|$. The method, applied in section 1, of repeating the process of backward recurrence for increasing starting values M of r till the computed values a_r remain constant in a desired number of significant figures may be considered correct therefore.

TABLE 2. Computed values for $a_r = (y_3)_r$ $(r \geq 0)$.

r	$\|a_r\| . 10^2$	arg a_r	r	$\|a_r\| . 10^2$	arg a_r
0	4.92	1.57	15	2.13	1.38
1	4.70	1.39	16	2.08	5.89
2	4.24	0.88	17	2.03	4.08
3	3.80	0.09	18	1.99	2.25
4	3.46	5.38	19	1.95	0.40
5	3.20	4.25	20	1.91	4.81
6	3.00	3.00	21	1.88	2.91
7	2.83	1.65	22	1.85	0.99
8	2.69	0.23	23	1.82	5.34
9	2.58	5.02	24	1.79	3.39
10	2.48	3.48	25	1.76	1.42
11	2.39	1.88	26	1.74	5.73
12	2.32	0.24	27	1.72	3.73
13	2.25	4.85	28	1.71	1.70
14	2.19	3.13			

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